Parametric roughness estimation in amplitude SAR images under the multiplicative model

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RESUMEN

En este trabajo se considera el problema de estimar la rugosidad de blancos sensados con radar de apertura sintética – SAR, bajo la hipótesis del Modelo Multiplicativo para datos en formato de amplitud, una o múltiples vistas y muestras de tamaño moderado. Se supone que los datos obedecen una distribución muy flexible, recientemente propuesta para la descripción de áreas con una gran variedad de rugosidad. Se presentan algunas propiedades de esta distribución, y diversos estimadores para sus parámetros son comparados utilizando una experiencia Monte Carlo. Se muestra una aplicación de los resultados en el filtrado de datos reales.

PALABRAS CLAVE: Estimación, Estadística, Model Multiplicativo, Radar de apertura sintética, Rugosidad.

ABSTRACT

The problem of estimating the roughness in SAR images is considered in this paper. It is assumed that this estimation is performed under the validity of the Multiplicative Model, for one- and multilook amplitude data and small-to-moderate sample sizes. Within the Multiplicative Model, a quite flexible distribution for the observations is considered, which has been recently proposed for the modelling of areas with a wide varying range of roughness. Some properties of this distribution are presented. Several estimators are compared using a Monte Carlo experience, and an application to filtering is provided.

KEY WORDS: Estimation, Multiplicative Model, Roughness, Synthetic aperture radar, Statistics.

INTRODUCTION

The precise knowledge of the statistical properties of synthetic aperture radar (SAR) data plays a central role in image processing and understanding. These properties can be used to discriminate types of land use and to develop specialised filters for speckle noise reduction, among other applications.

The multiplicative model has been widely used in the modelling, processing and analysis of synthetic aperture radar images. This model states that, under certain conditions, the response results from the product of the speckle noise by the terrain backscatter. Speckle noise, under very mild conditions, exhibits well known distributional properties. Several distributions could describe the backscatter, aiming at modelling different types of classes and their characteristic degrees of homogeneity. For instance, for some sensor parameters (wavelength, angle of incidence, polarisation etc.), a pasture area is more homogeneous than a forest, which, in turn, is more homogeneous than urban areas.

A successful statistical model for observations within homogeneous and heterogeneous areas is derived by multiplying two square root of Gamma-distributed random variables, a product that leads to the distribution for the amplitude data (other types of data formats are presented in Section 2). Though this distribution has proved to be very useful, it has some disadvantages, among which are:

- using it involves calculating the cumbersome modified Bessel function of the third kind;
- its cumulative distribution function can only be evaluated (in recursive form) when the restriction of an integer parameter is imposed;

- maximum likelihood estimators for its parameters are hard to obtain with the ease and precision required in may applications.

Many alternatives have been proposed in the literature, but most of them consist of abandoning the multiplicative model. This is not completely satisfactory, since the parameters of the distributions associated to the multiplicative model are fully interpretable in physical terms.

In this paper it is shown that a recent generalisation of the distributions associated to the multiplicative model yields a model with many advantages and interesting properties. It allows the modelling of extremely heterogeneous data beyond what K distributions could describe, and it is also flexible enough to fit data with an underlying K law. Besides these desirable characteristics, its analytic form is friendly since it does not involve the use of Bessel functions.

Some estimators for this model are derived and compared in this work. The importance of having good estimation techniques is detailed in the following sections, but it can be said here that it heavily relies on the fact that conclusions about the target roughness can be made through parameter estimation.

This paper is organised in the following manner: Section 2 (Image formation under the Multiplicative Model) presents the multiplicative model and the distributions that arise under this assumption; Section 3 (Estimation) recalls usual inference techniques (those based on maximum likelihood, moments and on order statistics), while Section 4 (Estimators for the considered distribution) specialises these techniques for the distribution here considered. The Monte Carlo experience designed to compare these estimators is presented in Section 5 (Estimators for the considered distribution), and the results of this experience are presented in Section 6 (Estimators for the considered distribution). Extensions and future work are commented in Section 7 (Estimators for the considered distribution).

IMAGE FORMATION UNDER THE MULTIPLICATIVE MODEL

This section is mainly based on the results presented in Correia et al (1998), Frery et al (1997a), Yanasse et. al (1995). The speckle noise is always associated to coherent-illuminated scenes, such as those obtained by microwaves, laser, ultrasound imaging etc. This kind of noise appears due to interference phenomena between the signals reflected by individual reflectors.

The multiplicative model is a common framework used to explain the stochastic behaviour of data obtained with coherent illumination. It assumes that the observations within this kind of images are the outcome of the product of two independent random variables X and Y, representing the terrain backscatter and the speckle noise, respectively (Tur, 1982). The former is frequently considered real and positive, while the latter can be complex (if the considered image is in complex format) or positive real (amplitude or intensity formats). In this work only amplitude format images will be considered.

Complex speckle, $\mathbf{Y}_c$, is usually assumed to have a bivariate normal distribution, with independent identically distributed components having zero mean and variance 1/2. Multilook amplitude speckle appears by taking the square root of the average of independent samples of the amplitude of $\mathbf{Y}_c$. In this manner the multilook amplitude speckle is given by $Y_A = \sqrt{\frac{1}{n}\left(\|Y_c\|_1 + \ldots + \|Y_c\|_n\right)}$, where $\|\mathbf{Y}_C\|_j$ denotes the j -th amplitude observation. This multilook amplitude speckle obeys a square root of Gamma distribution with unitary mean, denoted here as $\mathrm{Y}_A \sim \Gamma^{1/2}(n, n)$ characterised by the density

$$f_{Y_A}(y) = \frac{n^n}{\Gamma(n)} y^{2n-1} exp(-ny^2), n, y > 0. \qquad (1)$$

Fig. 1 shows three square root of Gamma densities, those corresponding to 1, 2 and 4 looks. It is noticeable that the bigger the more symmetric the density is. In fact, it is possible to prove that the variance, the skewness and the kurtosis of this distribution converge to zero when $n \to \infty$ (see Yanasse et al, 1995). This multilook processing is performed in order to reduce the influence of speckle noise in the image, but it leads to poorer spatial resolutions.

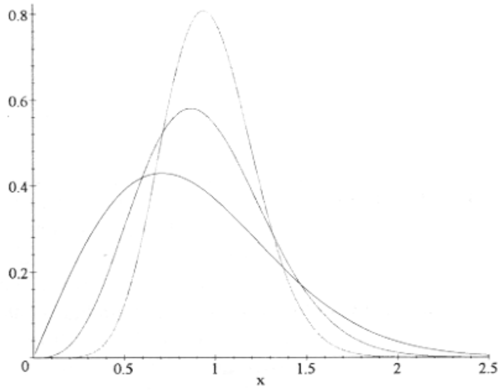


Figure 1. Speckle densities for 1, 2 and 4 looks (solid, dashes, dots resp.).

The backscatter may exhibit different degrees of homogeneity, and different models could be used to encompass this characteristic. Three main models have proved their usefulness in modelling amplitude backscatter: a constant (whenever the area is homogeneous to the sensor), a square root of Gamma distributed random variable (for heterogeneous areas) and, more recently, a square root of the reciprocal of a Gamma distributed random variable (for extremely heterogeneous areas). These three situations are unified by the square root of Generalised Inverse Gaussian distribution, whose density function is given by

$$f_{X_A}(x) = \frac{(\lambda/\gamma)^{\alpha/2}}{K_\alpha(2\sqrt{\lambda\gamma})} x^{2\alpha-1} exp\left(-\tfrac{\gamma}{x^2} - \lambda x^2\right), \qquad (2)$$

where $x > 0$, K_ν denotes the modified Bessel function of the third kind and order ν, and the parameters space is given by

$$\begin{cases} \gamma > 0, & \lambda \geq 0 \quad \text{if } \alpha < 0; \\ \gamma > 0, & \lambda > 0 \quad \text{if } \alpha = 0; \\ \gamma \geq 0, & \lambda > 0 \quad \text{if } \alpha < 0. \end{cases} \qquad (3)$$

The aforementioned particular cases are obtained imposing restrictions on the parameters space, and in order to make these derivations two properties of that Bessel function have to be recalled:

1. for every $x > 0$, $K_{-\nu}(x) = K_\nu(x)$, and
2. every $K_\nu(x)$ can be approximated by $\Gamma(\nu)2^{\nu-1} x^{-\nu}$ when $x \downarrow 0$ and $\nu > 0$.

These properties are also useful for specialising the moments provided in eq. (4) below.

The distribution induced by the density given in eq. (2) is denoted here as $X_A \sim N^{-1/2}(\alpha, \gamma, \lambda)$. Its r-th order moments are given by

$$\mathrm{E}(X_A^r) = (\gamma/\lambda)^{r/4} \frac{K_{\alpha+r/2}(2\sqrt{\gamma\lambda})}{K_\alpha(2\sqrt{\gamma\lambda})}. \qquad (4)$$

This distribution can be reduced to several important particular cases, but the following two are of special interest in applications:

- the square root of Gamma distribution, when $\gamma = 0$, denoted here as $\Gamma^{1/2}(\alpha, \lambda)$;
- the distribution of the reciprocal of a square root of Gamma distributed random variable, when $\lambda = 0$, denoted here as $\Gamma^{-1/2}(\alpha, \gamma)$.

The first special case leads to an *amplitude K* distribution for the response, when the speckle is square root of Gamma distributed (Yanasse et al, 1995). The second yields the model that will be discussed here. For detailed properties and applications of the square of the $N^{-1/2}(\alpha, \gamma, \lambda)$ the reader is referred to Barndorff-Nielsen and Blæsild (1981) and to Jørgensen (1982).

If $X_A \sim N^{-1/2}(\alpha, \gamma, \lambda)$ and $Y_A \sim \Gamma^{1/2}(n, n)$ are independent random variables, then the product $Z_A = X_A Y_A$ has a distribution which is called *amplitude G*. This distribution will be denoted here by $G_A(\alpha, \gamma, \lambda, n)$. It is characterised by the density

$$f_{Z_A}(z) = \frac{2n^n(\lambda/\gamma)^{\alpha/2}}{\Gamma(n)K_\alpha(2\sqrt{\gamma\lambda})} z^{2n-1}\left(\frac{\gamma+nz^2}{\lambda}\right)^{\frac{\alpha-n}{2}} \cdot K_{\alpha-n}(2\sqrt{\lambda(\gamma+nz^2)}), z > 0,$$

with and as in eq. (3).

This distribution for the amplitude response is quite general. As a counterpart, estimators for its parameters are very hard to obtain by maximum likelihood. In Frery et al (1997a) it was shown that a particular case, namely when $X_A \sim \Gamma^{-1/2}(\alpha, \gamma)$, leads to a special distribution for Z_A, denoted here as $G_A^0(\alpha, \gamma, n)$, with the following nice properties:

1. Its density only involves simple functions, since it is given by

   $$f_{Z_A}(z) = \frac{2n^n\Gamma(n-\alpha)z^{2n-1}}{\gamma^\alpha\Gamma(n)\Gamma(-\alpha)(\gamma+nz^2)^{n-\alpha}} \qquad (5)$$

   where $-\alpha, \gamma, n, z > 0$.

2. It allows the modelling of homogeneous, heterogeneous and extremely heterogeneous clutter. Specifically, data from deforested areas, from primary forest and from urban areas are very well fitted by this distribution.

3. Its cumulative distribution function is easily obtained, since the $G_A^0(\alpha, \gamma, n)$, distribution is readily seen to be proportional to the square root of the well-known Snedekor $F_{2n,-2\alpha}$ distribution (see Mejail, 1999 and Vasconcellos and Frery, 1998).

This last assertion can be justified recalling that a random variable W with a $\Gamma(\eta, 1/2)$ distribution is characterised by the density

$$f_W(w) = \frac{w^{\eta-1}exp(-w/2)}{2^\eta\Gamma(\eta)}, \eta, w > 0.$$

This density is that of a χ^2_{2n}-distributed random variable, therefore

$$-\frac{\alpha}{\gamma}Z_A^2 = \frac{2nY_A^2/(2n)}{2\gamma X_A'^2/(-2\alpha)} \sim F_{2n,-2\alpha},$$

where $X'_A = 1/X_A$, since the ratio of two independent a χ^2-distributed random variables yields this distribution. In this manner, the cumulative distribution function of a $G_A^0(\alpha, \gamma, n)$ random variable, which is given by

$$F_{Z_A}(t) = \Pr(Z_A \le t) = \frac{n^{n-1}\Gamma(n-\alpha)t^{2n}}{\gamma^n\Gamma(n)\Gamma(-\alpha)} H(n.n-\alpha; n+1; -nt^2/\gamma), \quad (6)$$

where H is the hypergeometric function (see Frery et al., 1997a), can be evaluated through the relation

$$F_{Z_A}(t) = \Upsilon_{2n,-2\alpha}(-\alpha t^2/\gamma), \quad (7)$$

where $\Upsilon_{\tau,v}$ is the cumulative distribution function of an $F_{\tau,v}$-distributed random variable. As F distribution arises in many important statistical problems, its cumulative distribution function is obtainable in a wide variety of statistical tables and systems. We have, then, an easy form of obtaining the cumulative distribution function for the G_A^0 distribution, by using standard statistical tables or routines for the F distribution.

The r-th order moment of a $G_A^0(\alpha, \gamma, n)$-distributed random variable (required to derive the moment estimators presented ahead) is given by

$$E(X_A^r) = \left(\frac{\gamma}{n}\right)^{r/2} \frac{\Gamma(-\alpha - r/2)\Gamma(n + r/2)}{\Gamma(-\alpha)\Gamma(n)}. \quad (8)$$

Some relationships among the aforementioned distributions for amplitude backscatter are summarised in Fig. 2, where $\xrightarrow{\bullet}$ denotes convergence in distribution when conditions "•" hold. The types of targets associated to each kind of return are also indicated: "Homog." ("Heterog" and "+Heterog", respectively) standing for homogeneous (heterogeneous and extremely heterogeneous, resp.) targets.

We assume in our study that the data are observations from independent identically distributed $G_A^0(\alpha, \gamma, n)$ random variables. The parameter is of particular interest in this work and in many applications, since it is directly related to the roughness of the target. This natural property, the roughness, if properly interpreted and quantified, may allow the discrimination among different types of classes, regardless of the power of the incident signal. The estimation of this sole parameter is the aim of this work.

The parameter γ is a scale parameter for the $G_A^0(\alpha, \gamma, n)$ distribution, and it is related to the relative power between reflected and incident signals. It may be estimated using extended targets, i.e., its estimator can be built using very large samples. Since the amount of information available for γ is usually extremely large in image applications, it can be assumed constant and known for all the small areas where α is being estimated. As γ is a scale parameter for this distribution, it can be chosen for every α of interest, without loss of generality, such that $E(Z_A) = 1$. In this manner, all the densities are comparable through the mean. Therefore, it is enough to study the inference of the parameter α for the $G_A^0(\alpha, \gamma_{\alpha,n}, n)$ distribution, where

$$\gamma_{\alpha,n} = n\left(\frac{\Gamma(-\alpha)\Gamma(n)}{\Gamma(-\alpha-1/2)\Gamma(n+1/2)}\right)^2$$

is assumed known and is the scale parameter that guarantees a unitary mean. The number of looks n is known, since it is fixed by the generation stage of the whole processing chain, and can be assumed integer.

Figs. 3 and 4 show the densities of the $G_A^0(\alpha, \gamma_{\alpha,n}, n)$ distribution for different values of the parameters. The former shows four densities corresponding to four equivalent number of looks n, namely $n \in \{1,2,4,8\}$, and for fixed $\alpha = -1.5$. It is observed that the symmetry of the return is also enhanced by

$$G_A^0(\alpha,\gamma,\lambda,n) \text{ General} \xrightarrow{\gamma\downarrow 0,\alpha,\lambda>0} K_A(\alpha,\lambda,n) \text{ Heterog.} \xrightarrow{\alpha,\lambda\to\infty,\alpha/\lambda\to\beta_1} \Gamma^{1/2}(n,n/\beta_1) \text{ Homog}$$

$$G_A^0(\alpha,\gamma,\lambda,n) \text{ General} \xrightarrow{\Lambda\downarrow 0,-\alpha,\Gamma>0} G_A^0(\alpha,\gamma,n) \text{ +Heterog.} \xrightarrow{\alpha,\gamma\to\infty,-\alpha/\gamma\to\beta_{21}} \Gamma^{1/2}(n,n\beta_2) \text{ Homog.}$$

Figure 2. Limiting properties of amplitude return distributions and associated targets.

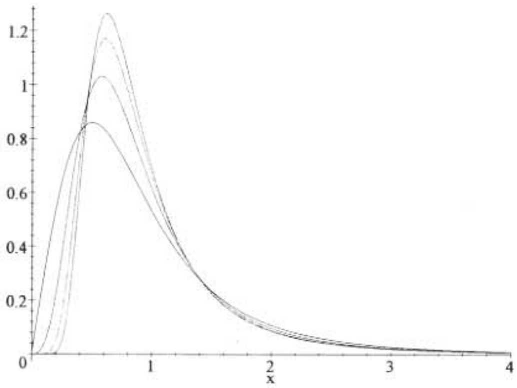


Figure 3. Four densities of the $G_A^0(\alpha,\gamma_{a,n},n)$ distribution for $n \in \{1,2,4,8\}$ and $\alpha = -1.5$ (solid, dash, dots, dot-dash respectively).

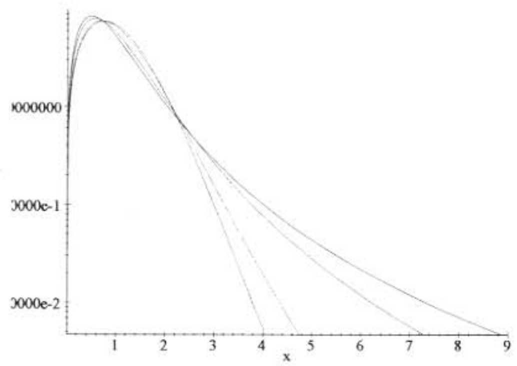


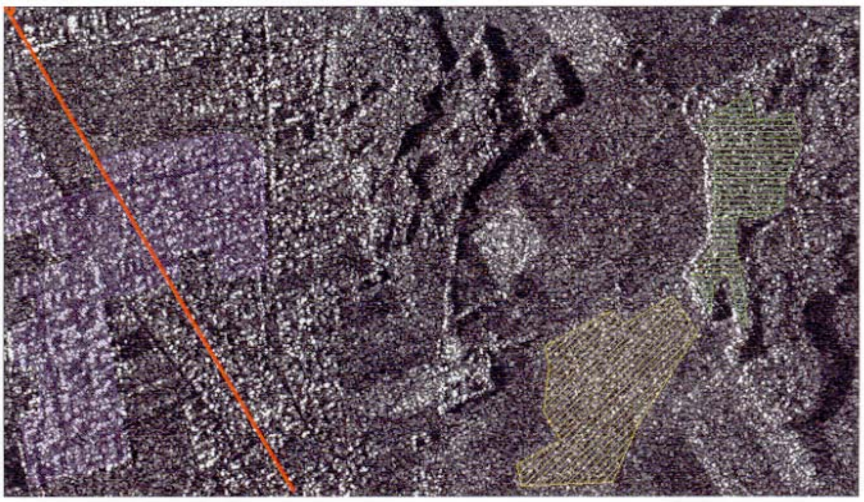
Figure 4. Four densities of the $G_A^0(\alpha,\gamma_{a,n},n)$ distribution for $-\alpha \in \{1.5,2,5,10\}$ and $n = 1$ (solid, dash, dots, dot-dash respectively) in semi-logarithmic scale.

the number of looks. Fig. 4 exhibits four densities corresponding to four values of the roughness parameter α, namely $-\alpha \in \{1.5,2,5,10\}$ and for fixed $n = 1$. This last plot is shown in semi-logarithmic scale, in order to enhance the fact that the bigger α is, the heavier the right-hand side tail of the density is. It can be seen that when $\alpha \geq -1/2$ the expected value is ∞. A detailed discussion of these and other properties of the G_A^0 distribution and its use in SAR image analysis can be found in Mejail, 1999.

Fig. 5 shows a 1-look ($n = 1$) amplitude SAR image obtained by the E-SAR airborne sensor over surroundings of München, Germany, originally of 1024×600 pixels. Several types of land use are visible in this image, markedly crops (where little or no texture is visible), forest (where there is some texture) and urban areas (where the texture is intense). The red slice is commented in the forthcoming sections.

Samples from these three classes will be used to show the adequacy of the distributions presented in Fig. 2: 26681 (15886 and 65650, respectively) pixels from crops (forest and urban area, resp.) are marked in yellow (green and purple, resp.) in the original area.

Three distributions were fit to these data sets, namely the $\Gamma^{1/2}$, K_A and G_A^0 distributions , through the estimation of their respective parameters. These estimated values (obtained by the methods presented in Frery et al., 1997a) are presented in Table 1, along with the p-value of the χ^2 test of each fit, between squared brackets. The result of fitting those distributions to the crops (forest and urban, respectively) data is shown in Figure 6 (7 and 8, resp.). From these figures and Table 1 it becomes evident that



* **Figure 5**. The original area under study, the analysed classes: crops (yellow), forest (green) and urban (purple) and the estimated slice (red).

Todas las figuras precedidas de asterisco se incluyen en el cuadernillo anexo de color.

Land Use	Sample Size	Colour	$\Gamma\,(1,1/\hat{\beta})$	$K_A\,(\hat{\alpha}, \hat{\lambda}, 1)$	$G_A^0(-\hat{\alpha}, \hat{\lambda}, 1)$
Crops	26681	Yellow	91156.5[.47]	(31.60,3.46 10^{-4})[.47]	(24.10,2.12 10^{6})[.50]
Forest	15886	Green	85550.6[.00]	(3.83,4.47 10^{-5})[.84]	(4.02,2.63 10^{5})[.66]
Urban Area	65650	Purple	271205.0[.00]	(0.36,1.34 10^{-6})[.00]	(1.20,7.65 10^{4})[.43]

Table 1. Main characteristics of the samples, estimated parameters and *p*-value of the χ^2 test (in brackets) for the three considered distributions.

- For homogeneous areas all the three distributions perform well (note that the difference between fitted densities is barely visible in Figure 6). The roughness parameters α of both K_A and G_A^0 distributions are, in absolute value, large suggesting convergence to the $\Gamma^{1/2}$ distribution.

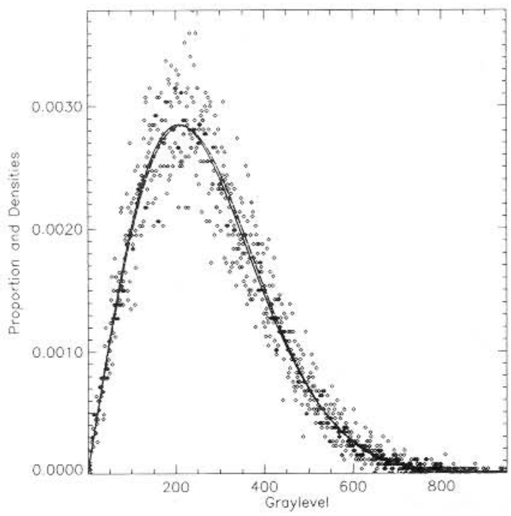


Figure 6. Crop data histogram and three fitted distributions: $\Gamma^{1/2}$ (dashes), K_A (long dashes) and G_A^0 (solid lines, respectively).

- For heterogeneous areas (Figure 7) the $\Gamma^{1/2}$ distribution produces a poorer fit than those provided by the other two distributions.
- When the area becomes extremely heterogeneous, as is the case of the urban area data presented in Figure 8, only the G_A^0 is capable of giving a good description of the observed values.

These conclusions, and the fact that the G_A^0 distribution does not require the use of Bessel functions, are the reasons why it has been proposed as a more general and more useful model than the K_A distribution for the fitting of SAR data (see Frery et al., 1997a and Mejail, 1999).

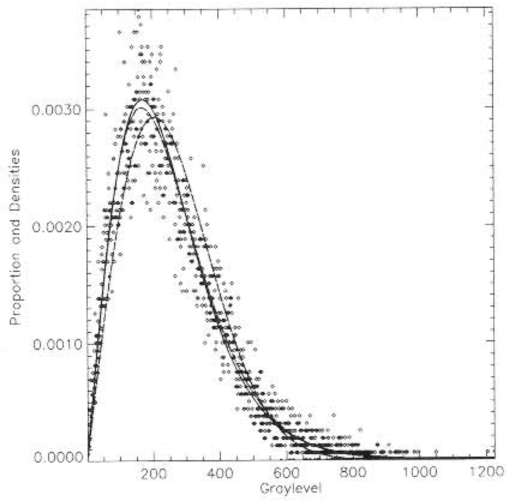


Figure 7. Forest data histogram and three fitted distributions: $\Gamma^{1/2}$ (dashes), K_A (long dashes) and G_A^0 (solid lines, respectively).

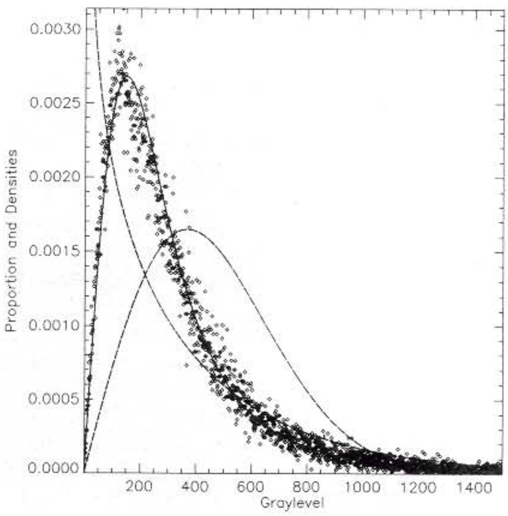


Figure 8. Urban data histogram and three fitted distributions: $\Gamma^{1/2}$ (dashes), K_A (long dashes) and G_A^0 (solid lines, respectively).

ESTIMATION

When using SAR images it is quite important to be able to assess the kind of land use in a precise and fast manner. Many techniques are available, but those based on statistical inference have proved their usefulness. The reader may check references Joughin et al (1993), Raghavan (1991), Silva et al. (1998) and Vasconcellos and Frery (1996, 1998) that deal with parameter estimation techniques assessment for SAR data distributions.

Among the advantages of the multiplicative model, there is one that relates the roughness to a single parameter: α. As it was presented in Section 2, it suffices to use the G_A^0 distribution for the modelling of every type of land use. It becomes evident, then, that estimating this parameter is an important task, since it allows quantitative roughness assessment within a single distributional model. As previously stated, we assume in this study that the data are observations from independent identically distributed G_A^0 (α, γ, n) random variables.

Several parameter estimation techniques are available, being the most remarkable ones those based on maximum likelihood (ML thereof), those based on sample moments or substitution method (MO for short), and those based on order statistics. These methods can be applied either to the original data or to the data after applying suitable transformations.

ML estimation is regarded as the *optimal* estimation technique since, for large samples, it can be proved that it yields unbiased and efficient (least variance among a large class) estimators. It may lead to too expensive numerical procedures (as is the case when dealing with *K* distributions), and the small sample properties of the estimators may be very hard to derive.

MO estimation is based on the substitution method, where theoretical moments (those presented in eq. (8), for instance) are replaced by sample moments, and unknown parameters are derived. This approach has the advantage of being quite simple to obtain, but it may be very hard to assess the properties of the derived estimators (especially for small sample sizes), besides the lack of unicity (an arbitrary number of estimators with different properties may be derived for the same parameter).

Estimators based on order statistics have been traditionally used to incorporate a degree of robustness in the inference procedure, but they may be hard to obtain and their sampling properties may be quite difficult to derive.

Several data transformations may lead to simple estimation techniques, and one of the most popular transformations of SAR data is the logarithm of the return. This popularity may be due to the fact that this operation, at least theoretically, transforms the multiplicative essence of the noise into a more familiar and tractable additive one. This approach suffers from its *ad hoc* or non-systematic nature.

In the following, MO, ML, and Order Statistics estimation techniques will be recalled.

Let $(Z_1, \ldots, Z_n)$ be a vector of independent identically distributed random variables, with distribution function F. Let f be a real function defined over $\mathbf{R}$ such that $E(|f(Z)|) < \infty$. Then $E(|f(Z)|)$ may be estimated by

$$\hat{m}_f = \frac{1}{N} \sum_{1 \le i \le N} f(Z_i) \qquad (9)$$

In eq. (9), if the function f is given by $z \mapsto z^r$ with $r > 0$, then we write $\hat{m}_r$ instead of $\hat{m}_z$. The parameters $\theta_1, \ldots, \theta_t$ can be, thus, estimated by a convenient set of equations of, for instance, the form

$$\begin{cases} E(Z_1)\,(\hat{\theta}_1,\ldots,\hat{\theta}_t) = \hat{m}_1 \\ \quad\ldots\ \ldots\ \ldots \\ E(Z_1^t)\,(\hat{\theta}_1,\ldots,\hat{\theta}_t) = \hat{m}t \end{cases}$$

Estimators obtained in this manner are asymptotically consistent and asymptotically jointly distributed as a multivariate normal random variable.

Now assume, without loss of generality, that F admits a density f. The ML method consists of maximising in the parameters $\theta_1, \ldots, \theta_t$ or finding the set of values $\hat{\theta}_1, \ldots, \hat{\theta}_t$ for which the likelihood function

$$L(\theta_1,\ldots,\theta_t \mid z_1,\ldots,z_N) \prod_{1 \le i \le N} f(z_i;\theta_1,\ldots,\theta_t)$$

is maximal. These estimators are usually found by explicit derivation of the likelihood or of its logarithm.

The ordered values of the vector $\mathbf{z} = (z_1,\ldots,z_N)$, i.e. $z_{N:1} \le \ldots \le z_{N:N}$, are called sample order statistics. It is intuitive to assume that, under certain very mild conditions, these values are comparable to the theoretical quantiles of the distribution $F(\theta_1, \ldots, \theta_t)$, which are given by

$$Q_a(\theta_1,\ldots,\theta_t) = \min_{z \in R}\{F(z;\theta_1,\ldots,\theta_t) \ge \alpha\}, \alpha \in (0,1)$$

In practice, the comparison is made between quartiles, i.e., between the lower quantile ($Q_{1/4}$), the median ($Q_{1/2}$) and the upper quartile ($Q_{3/4}$), or functions of these quartiles such as the Inter-quartile

Range ($Q_{3/4}$ - $Q_{1/4}$, IQR for short), and the corresponding sample quantities. The sample median of $\mathbf{z}$ is denoted as:

$$\hat{Q}_{1/2}(\mathbf{z}) = \begin{cases} z_{N:\lfloor N/2 \rfloor+1} & \text{if } N \text{ is odd} \\ \frac{1}{2}(z_{N:N/2} + z_{N:N/2+1}) & \text{if } N \text{ is even} \end{cases};$$

the lower sample quartile of $\mathbf{z}$ as:

$$\hat{Q}_{1/4}(\mathbf{z}) = \begin{cases} z_{N:(\ell+1)/2} & \text{if } \ell \text{ is odd} \\ \frac{1}{2}(z_{N:\ell/2} + z_{N:\ell/2+1}) & \text{if } \ell \text{ is even} \end{cases}; (10)$$

the lower sample quartile of $\mathbf{z}$ as:

$$\hat{Q}_{3/4}(\mathbf{z}) = \begin{cases} z_{N:(N+1-(\ell+1)/2} & \text{if } \ell \text{ is odd} \\ \frac{1}{2}(z_{N:N+1-\ell/2} + z_{N:N-\ell/2}) & \text{if } \ell \text{ is even} \end{cases}; (11)$$

where in eqs. (10) and (11) it was written

$$\ell \begin{cases} \frac{N-1}{2} & \text{if } N \text{ is odd} \\ \frac{N}{2} & \text{if } N \text{ is even} \end{cases}$$

For the sake of simplicity, the dependence of $\hat{Q}_\cdot$ on the sample $\mathbf{z}$ will be omitted whenever possible.

ESTIMATORS FOR THE CONSIDERED DISTRIBUTION

Three moment-based estimators will be used: those that use $\hat{m}_{1/2}$, $\hat{m}_1$ and $\hat{m}_1^{ln}$. This last one denotes the first moment of the logarithmicaly transformed data. Therefore, the three corresponding estimators $\hat{\alpha}_{1/2}$, $\hat{\alpha}_1$ and $\hat{\alpha}_1^{ln}$ are given by, respectively, the solutions of

$$\frac{\Gamma(-\hat{\alpha}_{1/2})}{\Gamma(-\hat{\alpha}_{1/2}-1/4)} = (\gamma/n)^{1/4} \frac{\Gamma(n-1/4)}{\hat{m}_{1/2}\,\Gamma(n)}, \qquad (12)$$

$$\frac{\Gamma(-\hat{\alpha}_1)}{\Gamma(-\hat{\alpha}_{1/2}-1/2)} = (\gamma/n)^{1/2} \frac{\Gamma(n-1/2)}{\hat{m}_1\,\Gamma(n)}, \qquad (13)$$

$$\Psi(-\hat{\alpha}_1^{\ln}) = 1n(\gamma/n) + \Psi(n) - 2\hat{m}_1^{\ln}, \qquad (13)$$

where Ψ is the digamma function. The maximum likelihood estimator $\hat{\alpha}_{\mathrm{ML}}$ is given by the solution of

$$\Psi(n-\hat{\alpha}_{\mathrm{ML}}) - \Psi(\hat{\alpha}_{\mathrm{ML}}) = -\ln\gamma + \frac{1}{N}\sum_{1\le i\le N}\ln(\gamma + nz_i^2). \quad (15)$$

Note that equations (12), (13), (14) and (15) are posed in the form $f(\hat{\alpha}_\bullet) = \tau$, in order to help the analysis of the estimators.

The estimators based on the median ($\hat{\alpha}_{\mathrm{Med}}$), on the inter-quartile range ($\hat{\alpha}_{\mathrm{IQR}}$) and on both of these order statistics ($\hat{\alpha}_{123}$) are, respectively, the solutions of

$$F_{Z_A}(\hat{Q}_{1/2};\hat{\alpha}_{\mathrm{Med}}) = 1/2, \qquad (16)$$

$$F_{Z_A}(\hat{Q}_{3/4};\hat{\alpha}_{\mathrm{IQR}}) - F_{Z_A}(\hat{Q}_{1/4};\hat{\alpha}_{\mathrm{IQR}}) = 1/2, \text{ and} \qquad (17)$$

$$F_{Z_A}(\hat{Q}_{3/4};\hat{\alpha}_{123}) - F_{Z_A}(\hat{Q}_{1/2};\hat{\alpha}_{123}) - F_{Z_A}(\hat{Q}_{1/4};\hat{\alpha}_{123}) = 0, \quad (18)$$

where F_{Z_A} is given in eq. (6), with the dependence on α made explicit. Using the result presented in eq. (7), estimating through eqs. (16), (17) and (18) reduces to solving, respectively

$$\Upsilon_{2n,-2\alpha_{\mathrm{Med}}}(\hat{Q}_{1/2}^2\hat{\alpha}_{\mathrm{Med}}/\gamma) = 1/2, \qquad (19)$$

$$\Upsilon_{2n,-2\alpha_{\mathrm{IQR}}}(\hat{Q}_{3/4}^2\hat{\alpha}_{\mathrm{IQR}}/\gamma) - \Upsilon_{2n,-2\alpha_{\mathrm{IQR}}}(\hat{Q}_{1/4}^2\hat{\alpha}_{\mathrm{IQR}}/\gamma) = 1/2, \qquad (20)$$

and

$$\Upsilon_{2n,-2\alpha_{123}}(\hat{Q}_{3/4}^2\hat{\alpha}_{123}/\gamma) - \Upsilon_{2n,-2\alpha_{123}}(\hat{Q}_{1/2}^2\hat{\alpha}_{123}/\gamma) - \Upsilon_{2n,-2\alpha_{123}}(\hat{Q}_{1/4}^2\hat{\alpha}_{123}/\gamma) = 0. \qquad (21)$$

The study of the two last estimators ($\hat{\alpha}_{\mathrm{IQR}}$ and $\hat{\alpha}_{123}$) will not be pursued in this study for reasons presented in the following section. Their definition remains, though, because estimators based on order statistics exhibit nice properties of robustness, and they can be used to build filters which are insensitive, to some extent, to deviations from the underlying hypothesis (see Frery et. al (1997b) for instance).

COMPARISON OF ESTIMATORS

A Monte Carlo study was performed in order to make a numerical comparison among the aforementioned estimators. This study consists of 10^5 replications for each situation where the sample size and parameters value change, followed by parameter estimation. In an exploratory stage $\hat{\alpha}_{1/2}$, $\hat{\alpha}_1$, $\hat{\alpha}_1^{\ln}$, $\hat{\alpha}_{\mathrm{ML}}$, $\hat{\alpha}_{\mathrm{Med}}$, $\hat{\alpha}_{\mathrm{IQR}}$, and $\hat{\alpha}_{123}$ were used, but the last two exhibited the poorest behaviour amongst the seven. In the results here presented, only the first five are, thus, compared.

The structure of the Monte Carlo experience is as follows

— Repeat for each replication $1 \leq i \leq 10^5$
  - For each number of looks $n \in \{1,2,4,8\}$, for each parameter $-\alpha \in \{1.5,2,5,10\}$ and for each sample size $m \in \{9,25,49,81\}$ do
    - Generate m independent samples of the $G_A^0(\alpha, \gamma_{\alpha,n}, n)$ distribution
    - Compute the five estimators $\hat{\alpha}_{1/2}$, $\hat{\alpha}_1$, $\hat{\alpha}_1^{\ln}$, $\hat{\alpha}_{ML}$, and $\hat{\alpha}_{Med}$ and using, for each, the available samples,
  - Calculate the mean, standard deviation and the mean square error of each estimator.

— Report results.

The results are presented in tabular form, i.e., for each triplet (n,α,m) the five estimators are numerically compared through their sample mean ($\bar{\alpha}_\bullet$), sample standard deviation ($s_\bullet$) and sample mean square error ($mse_\bullet = (\bar{\alpha}_\bullet - a)^2 + s_\bullet$). Each table presents the results for a certain value of n . No variance reduction techniques were employed. Selected plots are also presented.

The number of replications (10^5) was chosen in order to allow the assessment of the best estimator for the worst situation, namely, for the case $(n,\alpha,m) = (1,-1.5,9)$. Lewis and Orav (1989) propose techniques for changing the number of replications as a function of the variance of the quantity under study. This was not done, since CPU time was not a critical issue in this work.

The number of replications and the strong law of large numbers allow us to expect a negligible Monte Carlo variance, which is given by $10^{-5}\, s_\bullet^{\,2}$ for each estimator. There might remain doubts about the overall stability of the simulation procedure, due to possible instabilities in the simulation routines, for instance. In order to dissipate these doubts, the experience was repeated for the worst (with respect to the variance) case (i.e. for the values $(n,\alpha,m) = (1,-1.5,9)$) with a different seed, and the $mse_\bullet$ values were recorded. Since the obtained values are, for all the considered estimators, the same at least up to the third decimal place, and since only two decimal places are used in this study, it was considered unnecessary repeating the whole experience.

Fig. 9 illustrates how the data related to $\hat{\alpha}_{ML}$ behaves. The histogram to the right corresponds to the right-hand side of eq. (15); these data are summarised in the box-plot, where the sample mean is the big dot, $\hat{Q}_{1/2}$ is the bar close to it, surrounded by one sample standard deviation. The extreme bars denote $\hat{Q}_{3/4}$ and $\hat{Q}_{1/4}$, to the right and left respectively. These data are transformed, according to eq. (15), and the histogram of the resulting data is the one presented upside down. The accompanying box plot uses the same aforementioned codification.

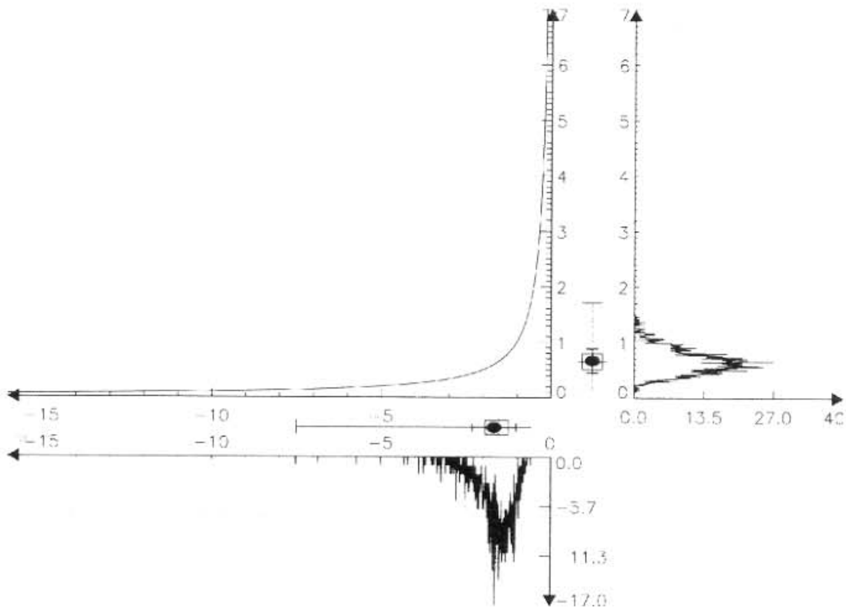


Figure 9. Counter-clockwise: histogram of the observed quantities, transformation function and histogram of estimated values for the $n = 1$, $\alpha = -1.5$ and $m = 9$ situation.

SIMULATION RESULTS AND APPLICATION

All the simulations were performed in a PC environment (Pentium 233 MHz, 64 MB RAM) using the IDL (http://www.rsinc.com) developing system for Windows 95. The generation of Gamma observations is based on the routine XG presented in Bustos and Frery, 1992. All seeds were recorded, in order to allow complete repeatability.

Tables 2, 3, 4 and 5 were compared using two criteria: the *mse* and the closeness to the true value. In the case of ties, for the first criterion, the closeness to the true value was used to break the ties choosing as "best" the closer one, i.e., the one with the least bias. For the second criterion, the sample standar deviation was used to break the ties choosing as "best" the smallest. Each particular situation will be referred to as the triplet of values (n,α,m).

Most of the time all the considered estimators underestimate the true value suffering, thus, consistently from a positive bias. The only exceptions to this observation are $\hat{\alpha}_{ML}$, (in 4,–1.5,81) , (8,–2,81) and (8,–10,81)) and $\hat{\alpha}_1^{\ln}$ (in (1,–1.10,9) , (2,–10,9) and (1,–10,25)). It is noteworthy that the bias of the former is quite smaller than the underestimation of the latter. The maximum likelihood estimator

$-\bar{\alpha}$	m	$-\bar{\alpha}_1(s_1)$	mse_1	$-\bar{\alpha}_{1/2}(s_{1/2})$	$mse_{1/2}$	$-\bar{\alpha}_1^{\ln}(s_1^{\ln})$	$mse_1^{\ln}$	$\bar{\alpha}_{ML}(s_{ML})$	mse_{ML}	$-\bar{\alpha}_{Q2}(s_{Q2})$	mse_{Q2}
1.5	9	1.72(0.79)	0.66	1.67(0.65)	0.46	1.67(0.75)	0.59	1.67(0.64)	0.44	1.76(1.06)	1.18
	25	1.58(0.33)	0.12	1.56(0.33)	0.11	1.56(0.38)	0.15	1.75(0.33)	0.11	1.58(0.48)	0.24
	49	1.54(0.23)	0.05	1.53(0.23)	0.05	1.53(0.26)	0.07	1.52(0.22)	0.05	1.54(0.33)	0.11
	81	1.53(0.18)	0.03	1.53(0.17)	0.03	1.52(0.20)	0.04	1.50(0.17)	0.03	1.53(0.25)	0.06
2	9	2.26(0.83)	0.77	2.22(0.87)	0.81	2.22(1.02)	1.08	2.23(0.85)	0.78	2.34(1.39)	2.05
	25	2.10(0.43)	0.20	2.08(0.45)	0.21	2.08(0.51)	0.27	2.07(0.44)	0.20	2.11(0.65)	0.44
	49	2.05(0.30)	0.09	2.04(0.31)	0.10	2.04(0.35)	0.12	2.03(0.30)	0.09	2.06(0.44)	0.19
	81	2.03(0.23)	0.05	2.02(0.24)	0.06	2.03(0.27)	0.07	2.01(0.23)	0.05	2.04(0.33)	0.11
5	9	5.56(2.13)	4.85	5.52(2.26)	5.36	5.39(2.36)	5.71	5.61(2.09)	4.73	5.87(3.40)	12.32
	25	5.18(1.09)	1.22	5.16(1.16)	1.37	5.17(1.33)	1.80	5.19(1.08)	1.21	5.27(1.61)	2.68
	49	5.10(0.76)	0.58	5.09(0.81)	0.66	5.10(0.93)	0.87	5.10(0.75)	0.57	5.15(1.09)	1.22
	81	5.06(0.58)	0.33	5.05(0.61)	0.38	5.06(0.70)	0.50	5.05(0.57)	0.33	5.09(0.83)	0.69
10	9	11.05(4.27)	19.33	10.99(4.55)	21.68	8.15(4.20)	21.11	10.77(3.11)	10.26	11.71(6.88)	50.28
	25	10.36(2.22)	5.05	10.34(2.37)	5.71	9.67(2.94)	8.74	10.38(2.08)	4.48	10.57(3.24)	10.85
	49	10.18(1.53)	2.38	10.16(1.64)	2.71	10.06(1.98)	3.91	10.19(1.49)	2.24	10.28(2.19)	4.86
	81	10.11(1.17)	1.37	10.11(1.25)	1.56	10.11(1.45)	2.10	10.12(1.13)	1.30	10.18(1.65)	2.75

Table 2. Simulation results for one look.

$-\alpha$	m	$-\bar{\alpha}_1(s_1)$	mse_1	$-\bar{\alpha}_{1/2}(s_{1/2})$	$mse_{1/2}$	$-\bar{\alpha}_1^{\ln}(s_1^{\ln})$	$mse_1^{\ln}$	$\bar{\alpha}_{ML}(s_{ML})$	mse_{ML}	$-\bar{\alpha}_{Q2}(s_{Q2})$	mse_{Q2}
1.5	9	1.66(0.51)	0.29	1.62(0.49)	0.25	1.60(0.50)	0.26	1.60(0.49)	0.25	1.63(0.67)	0.47
	25	1.56(0.32)	0.11	1.54(0.27)	0.07	1.54(0.28)	0.08	1.53(0.27)	0.07	1.55(0.37)	0.14
	49	1.53(0.20)	0.04	1.51(0.19)	0.04	1.52(0.20)	0.04	1.51(0.19)	0.04	1.53(0.26)	0.07
	81	1.52(0.15)	0.02	1.51(0.15)	0.02	1.52(0.15)	0.02	1.50(0.15)	0.02	1.53(0.20)	0.04
2	9	2.18(0.62)	0.41	2.14(0.62)	0.41	2.12(0.66)	0.44	2.14(0.63)	0.41	2.17(0.86)	0.77
	25	2.06(0.35)	0.12	2.05(0.35)	0.12	2.04(0.36)	0.13	2.03(0.35)	0.12	2.06(0.47)	0.22
	49	2.03(0.25)	0.06	2.02(0.24)	0.06	2.03(0.26)	0.07	2.01(0.24)	0.06	2.03(0.33)	0.11
	81	2.02(0.19)	0.04	2.01(0.19)	0.04	2.02(0.20)	0.04	2.00(0.19)	0.04	2.02(0.25)	0.06
5	9	5.30(1.43)	2.12	5.26(1.46)	2.20	5.23(1.54)	2.43	5.31(1.44)	2.16	5.37(1.96)	3.96
	25	5.10(0.80)	0.65	5.09(0.82)	0.68	5.08(0.87)	0.77	5.10(0.80)	0.65	5.13(1.09)	1.20
	49	5.05(0.56)	0.32	5.05(0.58)	0.33	5.05(0.61)	0.38	5.04(0.56)	0.32	5.07(0.76)	0.58
	81	5.03(0.43)	0.19	5.03(0.45)	0.20	5.03(0.47)	0.23	5.02(0.43)	0.19	5.04(0.58)	0.34
10	9	10.52(2.77)	7.96	10.45(2.85)	8.33	9.48(3.18)	10.41	10.50(2.49)	6.46	10.69(3.77)	14.72
	25	10.19(1.55)	2.43	10.16(1.60)	2.58	10.10(1.76)	3.10	10.20(1.53)	2.37	10.26(2.09)	4.43
	49	10.10(1.09)	1.20	10.08(1.13)	1.28	10.09(1.20)	1.45	10.10(1.08)	1.17	10.13(1.45)	2.13
	81	10.06(0.84)	0.71	10.05(0.87)	0.75	10.06(0.92)	0.86	10.05(0.83)	0.69	10.08(1.12)	1.26

Table 3. Simulation results for two looks.

$-\alpha$	m	$-\bar{\alpha}_1(s_1)$	mse_1	$-\bar{\alpha}_{1/2}(s_{1/2})$	$mse_{1/2}$	$-\bar{\alpha}_1^{\ln}(s_1^{\ln})$	$mse_1^{\ln}$	$\bar{\alpha}_{ML}(s_{ML})$	mse_{ML}	$-\bar{\alpha}_{Q2}(s_{Q2})$	mse_{Q2}
1.5	9	1.63(0.53)	0.30	1.59(0.42)	0.18	1.57(0.42)	0.18	1.57(0.42)	0.18	1.59(0.55)	0.31
	25	1.55(0.25)	0.07	1.54(0.24)	0.06	1.53(0.24)	0.06	1.52(0.24)	0.06	1.53(0.32)	0.10
	49	1.53(0.18)	0.03	1.52(0.17)	0.03	1.52(0.17)	0.03	1.50(0.17)	0.03	1.52(0.22)	0.05
	81	1.52(0.14)	0.02	1.51(0.13)	0.02	1.51(0.13)	0.02	1.49(0.13)	0.02	1.51(0.17)	0.29
2	9	2.14(0.52)	0.29	2.10(0.520	0.28	2.08(0.53)	0.29	2.08(052)	0.28	2.11(0.69)	0.48
	25	2.05(0.31)	0.10	2.04(0.30)	0.09	2.03(0.31)	0.09	2.02(0.30)	0.09	2.04(0.40)	0.16
	49	2.03(0.22)	0.05	2.02(0.21)	0.05	2.02(0.22)	0.05	2.00(0.21)	0.05	2.02(0.28)	0.08
	81	2.02(0.17)	0.03	2.01(0.16)	0.03	2.01(0.17)	0.03	2.00(0.16)	0.03	2.01(0.22)	0.05
5	9	5.19(1.09)	1.23	5.15(1.10)	1.23	5.13(1.12)	1.28	5.17(1.10)	1.24	5.19(1.41)	2.02
	25	5.07(0.63)	0.40	5.05(0.63)	0.40	5.05(0.65)	0.42	5.05(0.63)	0.40	5.07(0.45)	0.20
	49	5.03(0.45)	0.20	5.03(0.45)	0.20	5.03(0.46)	0.21	5.02(0.45)	0.20	5.04(0.58)	0.34
	81	502(0.35)	0.12	5.02(0.35)	0.12	5.02(0.36)	0.13	5.00(0.35)	0.12	5.02(0.45)	0.20
10	9	10.29(1.99)	4.05	10.25(2.01)	4.11	10.02(2.19)	4.80	10.29(1.95)	3.88	10.34(2.57)	6.73
	25	10.10(1.16)	1.36	10.09(1.17)	1.38	10.08(1.21)	1.47	10.09(1.15)	1.34	10.13(1.50)	2.26
	49	10.05(0.82)	0.67	10.04(0.83)	0.69	10.05(0.85)	0.73	10.04(0.81)	0.67	10.06(1.06)	1.12
	81	10.03(0.64)	0.40	10.03(0.64)	0.41	10.03(0.66)	0.44	10.02(0.63)	0.40	10.04(0.82)	0.67

Table 4. Simulation results for four looks.

$-\bar{\alpha}$	m	$-\bar{\alpha}_1(s_1)$	mse_1	$-\bar{\alpha}_{1/2}(s_{1/2})$	$mse_{1/2}$	$-\bar{\alpha}_1^{\ln}(s_1^{\ln})$	$mse_1^{\ln}$	$\bar{\alpha}_{ML}(s_{ML})$	mse_{ML}	$-\bar{\alpha}_{Q2}(s_{Q2})$	mse_{Q2}
1.5	9	1.62(0.44)	0.21	1.59(0.39)	0.16	1.57(0.39)	0.15	1.55(0.39)	0.15	1.57(0.50)	0.26
	25	1.55(0.24)	0.06	1.53(0.23)	0.05	1.53(0.22)	0.05	1.51(0.22)	0.05	1.53(0.29)	0.09
	49	1.55(0.17)	0.03	1.52(0.16)	0.03	1.52(0.16)	0.03	1.50(0.16)	0.03	1.51(0.21)	0.04
	81	1.52(0.13)	0.02	1.51(0.12)	0.02	1.51(0.12)	0.02	1.49(0.12)	0.02	1.51(0.16)	0.03
2	9	2.12(0.60)	0.38	2.09(0.47)	0.23	2.07(0.47)	0.23	2.06(0.47)	0.23	2.08(0.06)	0.37
	25	2.05(0.28)	0.08	2.03(0.28)	0.08	2.03(0.28)	0.08	2.01(0.28)	0.08	2.03(0.36)	0.13
	49	2.02(0.20)	0.04	2.02(0.20)	0.04	2.02(0.20)	0.04	2.00(0.20)	0.04	2.01(0.25)	0.07
	81	2.01(0.16)	0.03	2.01(0.15)	0.02	2.01(0.15)	0.02	1.99(0.15)	0.02	2.01(0.20)	0.04
5	9	5.14(0.91)	0.86	5.12(0.91)	0.85	5.10(0.92)	0.86	5.11(0.92)	0.85	5.13(1.15)	1.34
	25	5.05(0.54)	0.29	5.04(0.54)	0.29	5.04(0.54)	0.29	5.03(0.54)	0.29	5.05(0.68)	0.47
	49	5.30(0.38)	0.15	5.02(0.38)	0.15	5.02(0.39)	0.15	5.01(0.38)	0.15	5.02(0.49)	0.24
	81	5.02(0.30)	0.09	5.01(0.30)	0.09	5.02(0.30)	0.09	5.00(0.30)	0.09	5.02(0.38)	0.15
10	9	10.19(1.56)	2.48	10.16(1.57)	2.47	10.11(1.61)	2.60	10.17(1.56)	2.45	10.20(1.97)	3.91
	25	10.06(0.92)	0.85	10.05(0.92)	0.86	10.05(0.94)	0.88	10.04(0.92)	0.85	10.07(1.18)	1.39
	49	10.04(0.65)	0.43	10.03(0.65)	0.43	10.03(0.66)	0.44	10.01(0.65)	0.42	10.04(0.83)	0.69
	81	10.02(0.51)	0.26	10.01(0.51)	0.26	10.02(0.52)	0.27	9.99(0.51)	0.26	10.02(0.65)	0.42

Table 5. Simulation results for eight looks.

underestimates the parameter in situations where the sample size is small and the roughness is large, whereas the estimator based on the logarithmic transformation underestimates the parameter when there is high homogeneity and the number of looks is small.

For fifty-one cases in sixty-four, the $\hat{\alpha}_{ML}$ estimator was better than the other estimators were under the *mse* criterion. It was surpassed in only 6 situations for the following values: (1,–2,9) , (2,–5,9) and (2,–5,25) for which the $\hat{\alpha}_1$ estimator was the best one, and (1,–1.5,25), (2,–2,9) and (4,–5,9) for which the $\hat{\alpha}_{1/2}$ estimator had the best performance.

It can be seen that the estimator is not worse than the $\hat{\alpha}_{ML}$ estimator is for the following situations: and . Also, the $\hat{\alpha}_{ML}$ estimator is not worse than the estimator is for the following situations: (2,–1.5,49) , (4,–1.5,89) , (4,–15,25) , (8,–1.5,81) , (8,–2,81) and (8,–10.81) . So, the $\hat{\alpha}_{ML}$ estimator is not worse than the other estimators are in seven of the sixty-four situations.

This means that the $\hat{\alpha}_{ML}$ estimator was better than or equal to the other estimators according to the *mse* criterion in fifty-eight over sixty-four cases.

Using the closeness to the true value (bias) criterion, the $\hat{\alpha}_{ML}$ estimator is the best one in thirty-eight cases, and shares this best performance with other estimators in seven cases, over the sixty-four situations considered. The other nineteen situations correspond to the the following cases: (1,–10,81) where $\hat{\alpha}_1$ was the best one; 1.5,25), (1,–2,9), (1,–5,25), (1,–5,49), (2,–10,9) and (2,–10,49), where $\hat{\alpha}_{1/2}$ was the best one; and (1,–5,9), (1,-10,25), (1,–10,49), (2,–2,9), (2,–5,9), (2,–5, 25), (2,–10,25, (2,–5,9), (2,–10,9), (2,–10,25), (8,–5,9) and where $\hat{\alpha}_1^{\ln}$ was the best one.

For the following cases the behaviour of the $\hat{\alpha}_{ML}$ estimator was not worse than the behaviour of the other estimators: (2,–1,5,49), (4,–1.5,81), (4,–5,25), (8,–1.5,81), (8,–2,81) and (8,–10,81) for which it equalled the $\hat{\alpha}_{1/2}$ estimator; and (4,–1.5,9), (4,–1.5,81), (8,–1.5,81) and (8,–2,81), for which it equalled the $\hat{\alpha}_1^{\ln}$ estimator.

Summarising:

- Estimator based on the first moment $\hat{\alpha}_1$: There are three situations for which this is the best estimator with respect to the *mse* criterion, and only one situation that this happens with respect the closeness to the true value criterion.

- Estimator based on the half moment $\hat{\alpha}_{1/2}$: There are three situations for which this is the best estimator and six situations where it was equal to the $\hat{\alpha}_{ML}$ estimator, with respect to the $\hat{\alpha}_{ML}$ estimator, with resprct to the *mse* criterion. There are six situations for which this is the best estimator and six situations where it was equal to the $\hat{\alpha}_{ML}$ estimator, with respect to the closeness to the true value criterion.

- Estimator based on the first moment of the logarithmic data $\hat{\alpha}_1^{\ln}$: There are no situations, with respect to the *mse* criterion, for which this is the best estimator but there are four situations where it was equal to the $\hat{\alpha}_{ML}$ estimator. There are, with respect to the closeness to the true value criterion, twelve situations for which this is the best estimator and four situations where it was equal to the $\hat{\alpha}_{ML}$ estimator.

- Estimator based on the median $\hat{\alpha}_{Q2}$: This estimator performed poorly with respect to the criterion and, even though its mean value is close to the true value, its standard deviation is high.

Table 6 summarises the above description , where "w" stands for "win" and "t" for "tie". The criteria used are denoted "*mse*" for Mean Square Error and "*ctv*" for "Closeness to the True Value". The results are organised by the number of looks n.

		$n = 1$	$n = 2$	$n = 4$	$n = 8$	
		w/t	**w/t**	**w/t**	**w/t**	**w/t Total**
$\hat{\alpha}_1$	*mse*	1/0	2/0	0/0	0/0	3/0
	cts	1/0	0/0	0/0	0/0	1/0
$\hat{\alpha}_{1/2}$	*mse*	1/0	1/1	1/2	0/3	3/6
	ctv	4/0	2/1	0/2	0/3	6/6
$\hat{\alpha}_1^{\ln}$	*mse*	0/0	0/0	0/2	0/2	0/4
	cve	3/0	4/0	3/2	2/2	12/4
$\hat{\alpha}_{ML}$	*mse*	14/0	12/1	12/3	13/3	51/7
	ctv	8/0	9/1	10/3	11/3	38/7

Table 6. Comparison of estimators with the Mean Square Error (*mse*) and Closeness to the True Value (*ctv*) criteria.

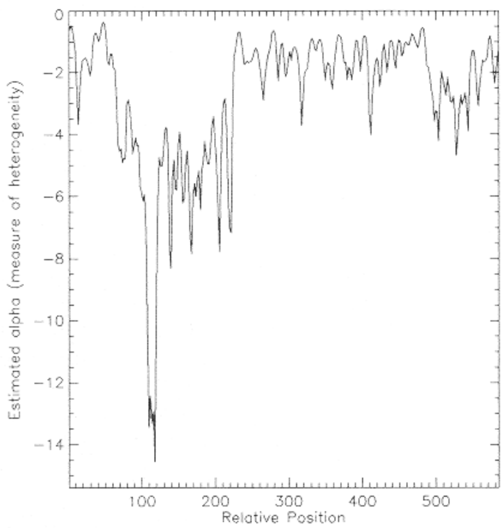


Figure 10. Estimated α within the red slice of the original image, using a 7×7 sliding window

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As an application, a slice of the original image was analysed with the $\hat{\alpha}_{ML}$ estimator. This slice is presented in red in Fig. 5, and it is noticeable that it runs through urban areas, interrupted by a more homogeneous region. The result of estimating α in this slice, using a sliding window of 7×7 pixels, is presented in Fig. 10. Reminding that α is a measure of heterogeneity, it becomes clear that this estimation is detecting both urban (extremely heterogeneous data, associated to small estimated values) areas and the forested region (less heterogeneous than the previous one and, thus, bigger estimated values).

CONCLUSIONS AND FUTURE WORK

In this paper different estimators were used for the estimation of the roughness parameter of the scaled G_A^0 distribution. A Monte Carlo study was performed and the $\hat{\alpha}_{ML}$ estimator was the best one in almost all cases.

In most of the considered situations, the $\hat{\alpha}_{ML}$ estimator is the best one with respect to both the and the closeness to the true value (bias) criteria.

The $\hat{\alpha}_{1/2}$ estimator was better than the $\hat{\alpha}_1$ estimator was for almost all of the looks and for both criteria.

The $\hat{\alpha}_1^{ln}$ estimator was closer to the true value than the $\hat{\alpha}_1$ estimator for all of the looks, and close than the $\hat{\alpha}_{1/2}$ for 2, 4 and 8 looks. With respect to the criterion, the $\hat{\alpha}_1^{ln}$ estimator is worse than the $\hat{\alpha}_1$ and the $\hat{\alpha}_{1/2}$ estimators are for all of the looks.

Taking into account that the computational effort required to compute $\hat{\alpha}_{ML}$ is comparable with the other estimators here studied, and considering its overall good performance, it can be concluded that it is the best choice for the problem at hand, even for small size samples.

In a future work, the simultaneous estimation of the α (roughness) and the γ (scale) parameters, will be considered.

BIBLIOGRAPHY

BARNDORFF-NIELSEN, O. E., BLÆSILD, P. B. 1981. *Hyperbolic Ddistributions and Ramifications: Contributions to Theory and Applications*. In: Taillie, C., Baldessari, B.A. *Statistical Distributions in Scientific Work*. Dordrecht, Reidel.

BUSTOS, O. H., FRERY, A. C. 1992. *Simulação Estocástica: Teoria e Algoritmos (versão completa)*. IMPA, Rio de Janeiro.

CORREIA, A. H., FREITAS, C. C., FRERY, A.C., SANT'ANNA, S. J. S. 1998. A user-friendly statistical system for polarimetric SAR image classification. *Revista de Teledetección*. 10:79-93.

FRERY, A. C.; MÜLLER, H. J.; YANASSE, C. C. F; SANT'ANNA, S. J. S. 1997a. A model for extremely heterogeneous clutter. *IEEE Trans. Geosc. Rem. Sens.* 35(3):648-659.

FRERY, A. C., SANT'ANNA, S. J. S., MASCARENHAS, N. D. A., BUSTOS, O. H. 1997b. Robust inference techniques for speckle noise reduction in 1-look amplitude SAR images. *Applied Signal Processing*. 4:61-76.

JØRGENSEN, B. 1982. *Statistical Properties of the Generalized Inverse Gaussian Distribution. Lecture Notes in Statistics*, vol. 9. Springer-Verlag, New York.

JOUGHIN, I. R., PERCIVAL, D. B., WINEBERNNER, D. P. 1993. Maximum likelihood estimation of K distribution parameters for SAR data. *IEEE Trans. Geosc. Rem. Sens.* 31(5):989-999.

LEWIS, P. A. W., ORAV, E. J. 1989. *Simulation Methodology for Statisticians, Operation Analysts and Engineers*, v. 1. Pacific Grove, CA, Wadsworth and Brooks/Cole.

MEJAIL, M. E. 1999. *La Distribución GA0 en el Modelado y Análisis de Imágenes de Radar de Apertura Sintética*. PhD Thesis in Computer Science, Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Argentina, 1999.

RAGHAVAN, R. S. 1991. A method for estimating parameters of K-distributed clutter. *IEEE Trans. Aer. Elec. Syst.* 27(2):238-246.

SILVA, L. B., VASCONCELLOS, K. L.P, FRERY, A. C. 1998. In: Second Latino-American Seminar on Radar Remote Sensing: Image Processing Techniques (SP 434), p. 45-48, Noordwijk, The Netherlands, 1996. European Space Agency.

TUR, M., CHIN, K. C., GOODMAN, J. W. 1982. When is speckle noise multiplicative? *App. Opt.* 21:1157-1159.

VASCONCELLOS, K. L. P., FRERY, A. C. 1998. Improving estimation for intensity SAR data. *InterStat*. 4(2):1-25.

VASCONCELLOS, K. L. P., FRERY, A. C. 1996. Maximum likelihood fitting of extremely heterogeneous radar clutter. 1997. In: First Latino-American Seminar on Radar Remote Sensing: Image Processing Techniques (SP 407), p. 97-101, Noordwijk, The Netherlands, 1996. European Space Agency.

YANASSE, C. C. F., FRERY, A. C., SANT'ANNA, S. J. S. 1995. *Stochastic Distributions and the multiplicative model: relations, properties, estimators and applications to SAR image analysis*. Technical Report 5630-NTC/318, INPE, São José dos Campos, SP, Brazil.